E-346

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

FACILITY FORM 602
N65-83559
(ACCESSION NUMBER)
13
(PAGES)
TMX 56288
(NASA CR OR TMX OR AD NUMBER)
(THRU)
None
(CODE)
(CATEGORY)

ON BODIES OF MINIMUM DRAG IN A SUPERSONIC GAS FLOW

By M. N. Kogan

Translation

"O telakh minimal 'nogo soprotivleniya v sverkhzvukovom potoke gaza." Prikladnaya matematika i mekhanika, t. XXI, no. 2, 1957, pp. 207-212.

TMX# 56288

Myron C. Nagurney
NASA Evaluator
4/27/65

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

ON BODIES OF MINIMUM DRAG IN A SUPERSONIC GAS FLOW

By M. N. Kogan

ABSTRACT

It is shown that the minimum drag problem for wings and bodies in supersonic flow can be treated through a determination of the potential and forces on the enclosing characteristic surfaces (Mach cones on plane Mach waves). An application of the method to a class of wings with straight edges is included.

INDEX HEADINGS

Flow, Supersonic 1.1.2.3

Wings, Complete - Theory 1.2.2.1

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

E-346

ON BODIES OF MINIMUM DRAG IN A SUPERSONIC GAS FLOW*

By M. N. Kogan

The problem of the determination of the minimum drag of wings and bodies in the supersonic flow of a gas is considered within the frame of the linear theory. In supersonic flow there are surfaces on which the forces acting on the bodies enclosed within these surfaces can be expressed through the values of the velocity potential on the surface. This condition permits reducing the problem of determining the extremal properties of three-dimensional bodies to a two-dimensional problem. In subsonic flow, in the general case, no such surfaces exist at a finite distance. As an example there is considered a wing with arbitrary supersonic leading edge and with trailing edge perpendicular to the flow. This problem is found to be entirely analogous to the classical problem of Munk (ref. 1) for a wing in an incompressible flow.

1. Fundamental theorem. - It is well known that the forces acting on a wing in an incompressible fluid can be expressed, in the infinitely removed plane, through the values of the velocity potential in this plane. Hence, in this case, it is possible to determine the minimum drag of the wing for a given lift force without going outside the Trefftz plane and without knowing the entire velocity field and shape of the wing having this drag.

Thus, in this case, as a result of the solution of the two-dimensional problem, first the forces and circulations over the wing span were determined and then the wing itself was obtained in order to simplify the problem.

In the supersonic case this procedure cannot be carried out in the infinitely removed plane.

By using the characteristic cone as control surface, A. A. Nikolski (ref. 2) split the problem of determining the minimum drag of a body of revolution in a supersonic flow.

In order to be able to split the variational problems, in the general case, it is necessary that on the chosen control surface the

*"O telakh minimal'nogo soprotivleniya v sverkhzvukovom potoke gaza." Prikladnaya matematika i mekhanika, t. XXI, no. 2, 1957, pp. 207-212.

forces acting on the bodies enclosed in it and the law of conservation of mass be expressed through the potential on that surface.

Let us consider a body or a group of bodies in the supersonic flow of a gas. The potential of the disturbed velocities is denoted by φ and the components of the velocities, respectively, on the x, y, and z axes are denoted by $u = \varphi_x$, $v = \varphi_y$, $w = \varphi_z$ (the subscripts denote differentiation).

The potential φ satisfies the equation

$$\beta^2\varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0 \qquad (\beta^2 = M_0^2 - 1) \tag{1.1}$$

where M_0 is the Mach number of the undisturbed flow. Let $F(x,y,z)$ be the equation of the control surface enclosing the bodies considered. The laws of conservation of mass and momentum can be written, respectively, in the form

$$\iint_F \frac{1}{N}\left[\beta^2\varphi_x F_x - \varphi_y F_y - \varphi_z F_z\right]ds = 0 \tag{1.2}$$

$$X = \rho_0 \iint_F \frac{1}{N}\left\{\frac{1}{2} F_x(\beta^2\varphi_x^2 + \varphi_y^2 + \varphi_z^2) - (\varphi_x\varphi_y F_y + \varphi_x\varphi_z F_z)\right\}ds \tag{1.3}$$

$$Y = \rho_0 v_0 \iint_F \frac{1}{N}(\varphi_x F_y - \varphi_y F_x)ds \tag{1.4}$$

$$\left(N = \sqrt{F_x^2 + F_y^2 + F_z^2}\right)$$

$$Z = \rho_0 v_0 \iint_F \frac{1}{N}(\varphi_x F_z - \varphi_z F_x)ds \tag{1.5}$$

where X, Y, Z are the components on the x, y, and z axes of the force applied to the bodies enclosed within the control surface and ρ_0 and v_0 are the density and velocity of the undisturbed flow, respectively.

Theorem. In order to express relations (1.2) to (1.5) through the values of the potential on F it is necessary and sufficient that the surface F be a characteristic surface of equation (1.1).

E-346

E-346

In fact, consider, for example, the expression in brackets in equation (1.2). We choose F in such manner that this expression represents a derivative along F.

Any derivative along F is proportional to the expression

$$\varphi_x F_z - \varphi_y y' F_z - \varphi_z (F_x + y' F_y) \tag{1.6}$$

where y' is an arbitrary direction. The expression in brackets in equation (1.2) is required to be proportional to equation (1.6), so that

$$\beta^2 \varphi_x F_x - \varphi_y F_y - \varphi_y F_z = A\left[\varphi_x F_z - \varphi_y y' F_z - \varphi_z (F_x + y' F_y)\right] \tag{1.7}$$

where A is the proportionality coefficient. Since this equation must be satisfied for any φ, it is equivalent to the following three conditions:

$$\beta^2 F_x = A F_z \qquad -F_y = A F_z y' \qquad F_z = A(F_x + F_y y') \tag{1.8}$$

Eliminating the arbitrary A and y', we obtain

$$\beta^2 F_x^2 - F_y^2 - F_z^2 = 0 \tag{1.9}$$

Thus F must satisfy equation (1.9) which, as is known, is the equation of the characteristics of the surfaces of equation (1.1).

The quadratic form in equation (1.3) is required to contain only the derivatives of φ along F. To satisfy this requirement it is necessary that

$$\frac{1}{2} F_x (\beta^2 \varphi_x^2 + \varphi_y^2 + \varphi_z^2) - (\varphi_x \varphi_y F_y + \varphi_x \varphi_z F_z)$$

$$= A(\varphi_x F_z - \varphi_z F_x)^2 + B(\varphi_x F_z - \varphi_z F_x)(\varphi_y F_z - \varphi_z F_y) + C(\varphi_y F_z - \varphi_z F_y)^2$$

Since φ is arbitrary, equation (1.10) is equivalent to the following six equations:

$$\frac{1}{2} F_x \beta^2 = AF_z^2 \qquad \frac{1}{2} F_x = CF_z^2 \qquad \frac{1}{2} F_x = AF_x^2 + BF_xF_y + CF_y^2$$

$$-F_y = BF_z^2 \qquad -F_z = -2AF_x - BF_zF_y \qquad BF_xF_z = -2CF_yF_z$$

The last equation of the six equations is a consequence of the second and fourth equations. The third and fifth equations, after eliminating A, B, and C, lead to equation (1.9), that is, the theorem has been also proven for equation (1.3).

The expressions under the integral signs in equations (1.4) and (1.5) are derivatives along F for any surface.

The theorem has thus been completely proven.

2. Wing of minimum drag for a given lift. - Let us consider an infinitely thin wing with arbitrary supersonic leading edge and trailing edge perpendicular to the flow (fig. 1). For the control surface we shall take the characteristic surface, passing through the perpendicular edge and the characteristic plane, passing through the trailing edge of the wing. Since the potential φ is an antisymmetric function of z, the flow can be considered only for $z \geq 0$.

For the lift Y and drag X according to equations (1.3) and (1.4) on the chosen control surface, we have

$$X = \frac{\rho_0}{M_0} \int_F\int [(\beta\varphi_x - \varphi_z)^2 + \varphi_y^2] ds$$

$$Y = \frac{2\rho_0 v_0}{M_0} \int_F\int (\beta\varphi_x - \varphi_z) ds \tag{2.1}$$

where the integration is carried out only on the after characteristic plane F, since $\varphi = 0$ on the forward characteristic surface.

The velocity $(\beta\varphi_x - \varphi_z)$ lies in the plane F.

The following characteristic variables are introduced:

$$\mu = x - \beta z \qquad \nu = x + \beta z \tag{2.2}$$

Equation (2.1) can then be written in the form

$$X = \frac{\rho_0}{2\beta}\iint_F \left[4\beta^2\varphi_\mu^2 + \varphi_y^2\right] dy\,d\mu \qquad Y = 2\rho_0 v_0 \iint_F \varphi_\mu\, dy\,d\mu \tag{2.3}$$

The problem consists in finding the minimum of the functional equation (2.3) for X with given Y. On the line $y = y(\mu)$ of intersection of the forward and after characteristic surfaces, $\varphi = 0$.

The Euler equation of this variational problem will evidently be

$$4\beta^2\varphi_{\mu\mu} + \varphi_{yy} = 0 \tag{2.4}$$

Integrating equation (2.3) by parts and introducing the new variables yields,

$$\xi = y \qquad \eta = \frac{l - \mu}{2\beta} = z \tag{2.5}$$

we obtain

$$\varphi_{\xi\xi} + {}_{\eta\eta} = 0 \tag{2.6}$$

$$X = -\rho_0 \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \varphi\varphi_\eta \big|_{\eta=0} d\xi \qquad Y = 2\rho_0 v_0 \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \varphi \big|_{\eta=0} d\xi \tag{2.7}$$

where l is the length of the wing and L the wing span (fig. 1).

Equations (2.6) and (2.7) agree with the equations obtained in the Trefftz plane for a wing in an incompressible fluid except that in the case considered, φ must become zero on the line $y = y(\mu)$ and not at infinity, as in the incompressible fluid.

It can easily be shown that, as in the case of the incompressible fluid, the functional for X from equation (2.7), for given Y, will be minimal if

$$\varphi_\eta \big|_{\eta=0} = C = \text{constant} \tag{2.8}$$

We introduce $\varphi_0 = \varphi/C$. The function φ_0 must satisfy the Laplace equation (2.6), becomes zero on the line $y = y(\mu)$ and $\varphi_{0\eta} = 1$ for $\eta = 0$. From equation (2.6) we then have

$$X = -\rho_0 C^2 A = -\frac{Y^2}{4\rho_0 v_0^2 A} \qquad Y = 2\rho_0 v_0 CA \qquad \left(A = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \varphi_0|_{\eta=0} d\xi\right) \quad (2.9)$$

The magnitude A, like φ_0, depends only on the shape of the control surface, which is determined by the shape of the wing.

We shall point out an interesting property of the flow around a wing of minimum drag.

On the after characteristic plane (that is, for $\nu = l$) the velocity $u = \varphi_x$ is constant along the line $\xi = y =$ constant. In fact, equation (1.1) valid for the entire flow in the coordinates μ, and ν, assumes the form:

$$4\beta^2 \varphi_{\mu\nu} - \varphi_{yy} = 0 \tag{2.10}$$

Comparing this equation with equation (2.4), valid on the plane $\nu = l$, we obtain

$$\frac{\partial}{\partial\mu}(\varphi_\mu + \varphi_\nu) = \frac{\partial}{\partial\mu}\varphi_x = u_\mu = 0 \tag{2.11}$$

that is, the assertion has been proven.

3. Numerical example. - We shall give a very simple numerical example. We consider a wing of minimum drag for which the potential in the plane $\nu = l$ has the form

$$\varphi = a\xi^2 + b\eta^2 + c\eta + d \tag{3.1}$$

where a, b, c, and d are constants.

In order that the wing possess a minimum drag for a given lift, φ must satisfy the Laplace equation (2.6). For this reason, we must have

$$a = -b = k = \text{constant}$$

For $\xi = \pm\frac{1}{2} L$ and $\eta = 0$ we must have

$$\varphi = \frac{1}{4} kL^2 + d = 0 \quad \text{or} \quad d = -\frac{1}{4} kL^2$$

Further, since at the point $\nu = l$, $\xi = 0$ and $\mu = l - 2\beta\eta = 0$ the potential is equal to zero, so that

$$c = k \frac{\left(\frac{1}{2} L\right)^2 + \left(\frac{\frac{1}{2} l}{\beta}\right)^2}{\frac{\frac{1}{2} l}{\beta}}$$

Hence,

$$\varphi = k\left[\xi^2 - \eta^2 + \frac{\left(\frac{1}{2} L\right)^2 + \left(\frac{\frac{1}{2} l}{\beta}\right)^2}{\frac{\frac{1}{2} l}{\beta}} \eta - \left(\frac{L}{2}\right)^2\right] \tag{3.2}$$

Since $\varphi = 0$ on the line $\xi = y = y(\mu) = y_1(\eta)$, the line is a hyperbola

$$\xi^2 - \eta^2 + \frac{\left(\frac{1}{2} L\right)^2 + \left(\frac{\frac{1}{2} l}{\beta}\right)^2}{\frac{\frac{1}{2} l}{\beta}} \eta - \left(\frac{L}{2}\right)^2 = 0 \tag{3.3}$$

Since the forward characteristic surface is an envelope of the Mach cones issuing from the points of the leading edge (if the line of intersection of this envelope with the plane $\nu = l$ is known) the shape of the leading edge in parametric form can easily be found:

$$\frac{x}{l} = 1 - 2\lambda\left[\frac{\lambda - \frac{1}{4}(L^{*2} + 1)}{\frac{1}{4}(L^{*2} - 1)}\right] \tag{3.4}$$

$$\frac{y}{l} \beta = \sqrt{\lambda^2 - \frac{1}{2} \lambda(L^{*2} + 1) + \frac{1}{4} L^{*2}}\left\{1 - \frac{32\lambda\left[\lambda - \frac{1}{4}(L^{*2} + 1)\right]}{(L^{*2} - 1)^2}\right\} \tag{3.5}$$

where $L^* = L\beta/l$ and λ is a parameter.

Figure 2 presents the shape of the wings both for various values of L^* and for the corresponding lines of intersection of the characteristic surfaces. For each wing shape the values of $L\beta/2l$ are indicated on the curves; for $L^*/2 < \sqrt{5/4}$ the leading edge becomes subsonic (the dotted lines in fig. 2 correspond to a delta wing with sonic edge).

Substituting equation (3.2) into equation (2.11), we obtain

$$X = \frac{3}{8}\,\frac{\left[\left(\frac{1}{2}\right)L^{2} + \left(\frac{\frac{1}{2}\,l}{\beta}\right)^{2}\right]}{\rho_0 v_0 \left(\frac{1}{2}\right)L^{2}\left(\frac{l}{\beta}\right)}\,Y^2 \tag{3.6}$$

Figure 3 gives the ratio X/Y^2 for the wing of minimum drag to the corresponding magnitude for the plane wing of the same plan form.

4. Remarks. - 1. The preceding sections show how the forces acting on a body of minimum drag are determined. However, to determine the shapes of the bodies that produce these forces it is necessary to solve the very complicated (in the general case), three-dimensional problem of Goursat of constructing the field from the data on the characteristic surfaces. Practically however it is advantageous not to solve this problem but to select, among wings of relatively simple shape, those which possess a drag approximating the extremal value. Thus, in figure 3 a curve has been plotted corresponding to a delta wing with conical twist, given according to the law

$$\alpha(x,\ y) = a + b(y/x)^2$$

(α is the local angle of attack and a and b are constants). We see that the aerodynamic characteristics of this wing approximate those of the extremal.

Analogous considerations show that it is not necessary to be concerned with the absence of additional bodies within the flow. The obtained minimum drag must be considered as a lower limit which, as the examples have shown, is attainable.

2. According to equation (2.11) in the plane $\nu = l$ the velocity u is a function only of y. If φ is known, in the plane $\nu = l$, the function $u(y)$ on the line of intersection of the forward and after

characteristic surfaces can easily be found. Reference 3 shows for the integrals across the flow

$$\overline{u} = -\frac{\overline{w}}{\beta}$$

$$\left(\overline{u} = \int u \, dy, \ \overline{w} = \int w \, dy\right) \tag{4.1}$$

E-346

and that $\overline{u}$ = constant and $\overline{v}$ = constant along the characteristics μ = constant. Knowing $u(y)$ in the plane $\nu = l$ and taking account of the fact that $\overline{u}$ = constant along μ = constant the lift force distribution along the wing chord is easily obtained.

3. Up to now we have considered wings with supersonic edges. However, all that has been said also remains valid for wings with subsonic edges. In this case the region of integration will be cut out from the plane $\nu = l$ by the Mach cone issuing from the leading edge of the wing. In the plane $\nu = l$ on the line $\eta = 0$ it is necessary, in determining φ, to put $\varphi = 0$ outside the wing. If we consider a wing of very small aspect ratio $(L/l \ll 1)$, the condition of φ becoming zero on the line $y = y(\mu)$ will be equivalent to having the potential become zero at infinity, as for an incompressible fluid. It is evident that in this case the extremal will be a wing with elliptical spanwise load distribution, as for an incompressible fluid.

However, the flow producing an extremal wing in the supersonic case will differ essentially from the corresponding flow in the incompressible case. In fact, according to equation (2.11), the velocity u in the plane $\nu = l$ is constant along the line y = constant. Hence the velocity μ will be finite also on the forward characteristic cone (the velocity u can not be equal to zero on the trailing edge because according to eq. (2.11) it would be equal to zero on the entire plane $\nu = l$, while according to eq. (4.1), in this case, the lift would be equal to zero). It is known, however, from the theory of characteristics that a finite discontinuity on the characteristic cone is obtained only in the presence of infinite disturbances in its vertex.

Thus, the wing can possess a blunt vertex similar to the so-called minimal-resistance Kármán cap. The method is thus also applicable to the study of the extremal properties of wings with subsonic edges. However, the value of such investigations is limited because the forces determined by the theorem of conservation of momentum also include the suction force. Experience shows, however, that the magnitude of the suction force obtained by the linear theory is generally not realized. Hence, the extremal properties of wings with subsonic edges represent only an ideal lower limit.

REFERENCES

1. Munk: Isoperimetrische Aufgaben aus der Theorie des Fluges. Dissertation, Göttingen, 1919. (See Durant, vol. II.)

2. Nikolski, A. A.: On Ducted Bodies of Revolution Possessing Minimal Internal Drag in Supersonic Flow. Trudy TsAGI, 1950.

3. Kogan, M. N.: Certain Properties of Supersonic Flows. PMM, no. 5, 1956. (See also Trudy TsAGI, no. 687.)

E-346

Translated by S. Reiss
National Aeronautics and
Space Administration

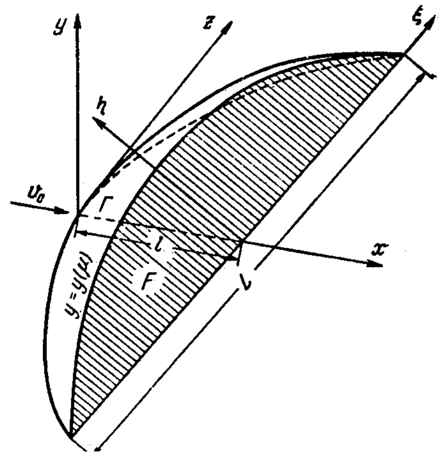

Figure 1.

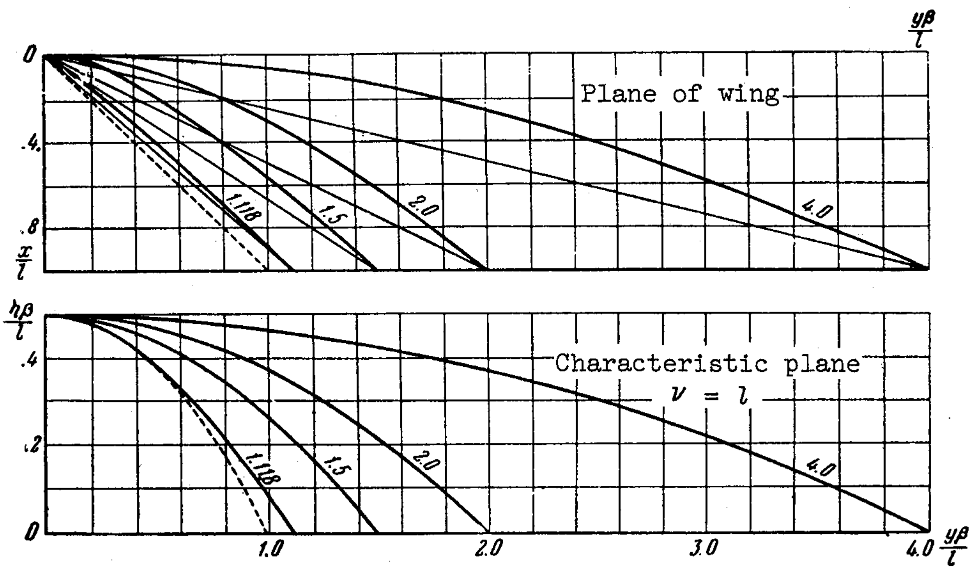

Figure 2.

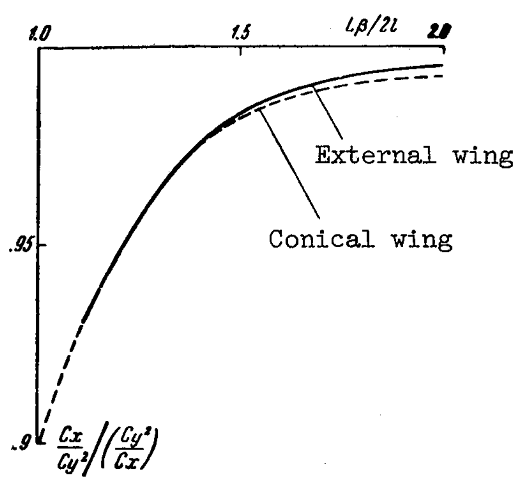

Figure 3.